The heat equation and Bochner formula

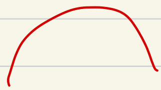
1. Heat equation and Harnack inequality
2. Brief introduction to Riemannian geometry
3. Bochner formula and heat eqn on manifolds.

Laplacian: $$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$f(x) = x^2$

$\Delta f(x) > 0$

$f(x) = -x^2$



$\Delta f(x) < 0$

$f(x,y) = x^2 - y^2$

$\Delta f(x) = 0$. harmonic functions.

Heat eqn: $$\partial_t u(x,t) = \Delta u(x,t)$$

IVP $$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & \text{for } t \geq 0. \\ u(x,0) = u_0(x) & \end{cases}$$

← initial configuration of heat

Properties of the heat equation:

1) Heat equation is linear

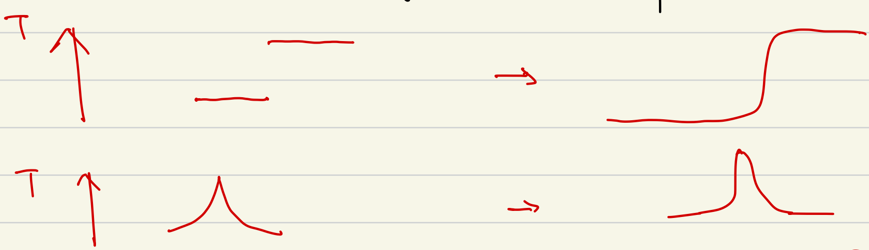
2) Can always solve the IVP for $t > 0$.

3) Total heat is preserved $\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} \Delta u(x,t)\,dx = 0$

4) Tend to "smooth" out the initial data

5) Instantaneous response at any distance.

4)



5)

These properties are often quantifiable.

Thm: (Weak version of 4)

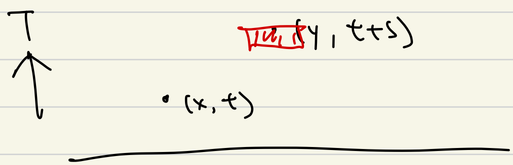
1) $\sup_M u(x,t) \le \sup_M u(x,0)$, $\inf_M u(x,t) \ge \inf_M u(x,0)$

2) If $|\nabla^k u(x,0)| \le C \Rightarrow |\nabla^k u(x,t)| \le C$

Thm: (Harnack inequality)

If $u \geq 0$, $\forall R > 0$, $s > 0$, $\exists \varepsilon = \varepsilon(R, s, t)$ s.t

$$u(y, t+s) \geq \varepsilon u(x, t) \quad \text{for } |x-y| \leq R$$



Corollary: (Strong max principle)

If $u \geq 0$ is a sol of the heat eqn and

$u(x, t) = 0$ for some $t > 0$, $\Rightarrow u \equiv 0$

Thm: (Weak version of 4)

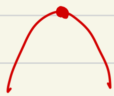
1) $\sup_M u(x,t) \le \sup_M u(x,0)$, $\inf_M u(x,t) \ge \inf_M u(x,0)$

2) If $|\nabla^k u(x,0)| \le C \implies |\nabla^k u(x,t)| \le C$

"Pf of thm":

1) Look at x_0 s.t $u(x_0,t) = \sup_M u(x,t)$

$\implies$ at that point



$$\Delta u(x_0,t) \le 0$$

$$\Rightarrow \quad \partial_t u(x_0,t) = \Delta u(x_0,t) \le 0.$$

2) $\partial_t u = \Delta u \implies \dfrac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_n}}$ also solves the heat eqn

apply 1)

new proof:

$$\partial_t |\nabla u|^2 = \frac{\partial}{\partial t}\left(\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right)$$

$$= \sum_i 2 \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial t}$$

$$= \sum_i 2 \frac{\partial u}{\partial x_i} \frac{\partial(\Delta u)}{\partial x_i}$$

$$= \sum_i 2 \frac{\partial u}{\partial x_i} \Delta \frac{\partial u}{\partial x_i}$$

new proof: of $k=1$.

$$\partial_t |\nabla u|^2 = \frac{\partial}{\partial t}\left(\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)$$

$$= \sum_i 2\,\frac{\partial u}{\partial x_i}\,\frac{\partial^2 u}{\partial x_i \partial t}$$

$$= \sum_i 2\,\frac{\partial u}{\partial x_i}\,\frac{\partial(\Delta u)}{\partial x_i}$$

$$= \sum_i 2\,\frac{\partial u}{\partial x_i}\,\Delta\frac{\partial u}{\partial x_i}$$

$$= \Delta|\nabla u|^2 - \underbrace{\sum_{i,j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2}_{\leq 0}$$

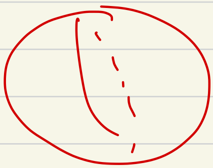
subsolution

$$\Rightarrow \partial_t |\nabla u|^2 \leq \Delta |\nabla u|^2$$

By maximum principle $\Rightarrow$ bounds on $|\nabla u|^2$ are preserved.

Riemannian manifold:

Ex:



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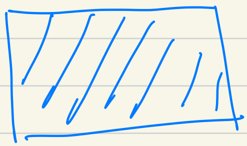

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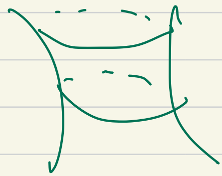

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local invariant $\longrightarrow$ Riemann curvature.

$\longrightarrow$ Ricci curvature. $\left(Ric(X, Y) = \text{number} \right)$

Heat equation makes sense on a Riem mfd.

Q: Can we get the same estimates for solution of the heat equation on a manifold?

Bochner formula: u any function on M

$$(*) \qquad \boxed{\frac{1}{2} \underbrace{\Delta |\nabla u|^2}_{\Delta \text{ term}} = \underbrace{|\nabla^2 u|^2}_{\geq 0} + \nabla(\underbrace{\Delta u}_{\Delta \text{ term}}) \cdot \nabla u + Ric(\nabla u, \nabla u)}$$

Thm: (Weak version of 4)

1) $\sup_M u(x,t) \le \sup_M u(x,0)$, $\inf_M u(x,t) \ge \inf_M u(x,0)$ $\Rightarrow$ always true

2) If $|\nabla u(x,0)| \le C \Rightarrow |\nabla u(x,t)| \le C$

Q: When is ↑ true on M, for solutions of the heat equation

For property 2)

$$\frac{\partial}{\partial t}|\nabla u|^2 = 2 \; \nabla\left(\frac{\partial}{\partial t}u\right)\cdot \nabla u$$

$$= 2 \; \nabla(\Delta u)\cdot\nabla u$$

Bochner formula

$$= -2|\nabla^2 u|^2 + \Delta|\nabla u|^2 - \mathrm{Ric}(\nabla u, \nabla u)$$

$$= \Delta|\nabla u|^2 \underbrace{- 2|\nabla^2 u|^2}_{\le 0} \underbrace{- \mathrm{Ric}(\nabla u,\nabla u)}_{\text{if } \mathrm{Ric}\ge 0 \text{ then this term is negative}}$$

If $\mathrm{Ric} \ge 0$, then $\partial_t u = \Delta u \Rightarrow \partial_t |\nabla u|^2 \le \Delta|\nabla u|^2$

$\Rightarrow$ max principal 2) holds.

In general 2) doesn't hold.

$$\frac{\partial}{\partial t}|\nabla u|^2 = 2 \quad \nabla\left(\frac{\partial}{\partial t}u\right)\cdot\nabla u$$

$$= 2 \quad \nabla(\Delta u)\cdot\nabla u$$

Bochner formula

$$= -2|\nabla^2 u|^2 + \Delta|\nabla u|^2 - \mathrm{Ric}(\nabla u, \nabla u)$$

$$= \Delta|\nabla u|^2 \underbrace{- 2|\nabla^2 u|^2}_{\leq 0} \underbrace{- \mathrm{Ric}(\nabla u, \nabla u)}_{\substack{\text{if } \mathrm{Ric} \geq -Kg \\ \text{then } \leq K|\nabla u|^2}}$$

If $\underline{\mathrm{Ric} \geq -K}$, then $\partial_t u = \Delta u$

$$\Rightarrow \partial_t|\nabla u|^2 \leq \Delta|\nabla u|^2 + K|\nabla u|^2$$

$$\Leftrightarrow \partial_t\left(e^{2Kt}|\nabla u|^2\right) \leq \Delta\left(e^{2Kt}|\nabla u|^2\right)$$

$$\Rightarrow \text{If } |\nabla u(x,0)|^2 \leq C \Rightarrow |\nabla u(x,t)|^2 \leq e^{2Kt} C$$

In fact the $\Rightarrow$ characterizes the condition $\mathrm{Ric} \geq -K$

that is for a cpt Riemannian manifold (M, g)

$\mathrm{Ric} \geq -K \quad \Leftrightarrow$ for any $u(x,t)$ satisfy $\partial_t u = \Delta u$

one has $\partial_t|\nabla u|^2 \leq \Delta|\nabla u|^2 + K|\nabla u|^2$

Li-Yau Harnack inequality:

Suppose $(\partial_t - \Delta) u(t,x) = 0$, and $u \geq 0$

$$\Delta |\nabla \log u|^2 \geq |\nabla^2 \log u|^2 + \langle \nabla \underbrace{\Delta \log u}, \nabla \log u \rangle$$

plug $f = \log u$ into Bochner formula

$$\Delta \log u = \frac{\Delta u}{u} - |\nabla \log u|^2$$
$$= \partial_t \log u - |\nabla \log u|^2$$

set $f = \log u$, then $(\partial_t - \Delta) f = |\nabla f|^2$

$$\Rightarrow \Delta |\nabla f|^2 \geq |\nabla^2 f|^2 + \langle \nabla \Delta f, \nabla f \rangle$$
$$\geq \frac{1}{n} (\Delta f)^2 + \langle \nabla \Delta f, \nabla f \rangle$$

$$\Delta(\partial_t f - \Delta f) \geq \frac{1}{n} (\Delta f)^2 + \langle \nabla \Delta f, \nabla f \rangle$$

$$\boxed{(\partial_t - \Delta)(\Delta f) \geq \frac{1}{n} (\Delta f)^2 + \langle \nabla \Delta f, \nabla f \rangle} \Rightarrow -\Delta f \leq \frac{n}{2t}$$

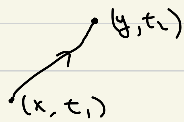
by maximum principle

$$\Rightarrow \boxed{\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} \geq -\frac{n}{2t}}$$

(note $f = -\frac{n}{2t}$ solves $(\partial_t - \Delta) f = \frac{1}{n} f^2$)

$0 \leq t_1 < t_2$

integrate along



$$\Rightarrow u(x, t_1) \leq u(y, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t_2 - t_1)}}$$

"sharp Harnack inequality"

Bochner Formula:

$$\boxed{\underbrace{\frac{1}{2}\Delta|\nabla u|^2}_{\text{Laplacian}} = \underbrace{|\nabla^2 u|^2}_{\geq 0} + \langle \nabla \underbrace{\Delta}_{\text{Laplacian}} u, \nabla u\rangle + \underbrace{Ric(\nabla u, \nabla u)}_{\text{Ricci curvature}}}$$

can be controlled if curvature is bounded

Corollary 1: If $Ric \geq 0$, and $\Delta u = 0$ then

$$\boxed{\frac{1}{2}\Delta|\nabla u|^2 \geq 0} \rightsquigarrow$$ Cheng-Yau gradient est and Liouville theorem.

Corollary 2: If $Ric \geq 0$ and $\partial_t u = \Delta u$, then

$$\boxed{\frac{1}{2}\left(\Delta|\nabla u|^2 - \partial_t|\nabla u|^2\right) \geq 0} \longrightarrow$$ Bakry-Emery estimates

Corollary 3: If $Ric \geq 0$ and $|\nabla u|^2 = \text{const}$

$$\Rightarrow \boxed{\begin{aligned} 0 &= |\nabla^2 u|^2 + \nabla_{\nabla u}(\Delta u) + Ric(\nabla u, \nabla u) \\ &\geq \frac{1}{n}(\Delta u)^2 + \nabla_{\nabla u}(\Delta u)\end{aligned}} \longrightarrow$$ Laplace comparison and Bishop-Gromov inequality

Corollary 4: If $Ric \geq 0$, $|\nabla u| = \text{const}$ and $\Delta u = 0$

$$\boxed{\Rightarrow |\nabla^2 u| \equiv 0.} \rightsquigarrow$$ Cheeger-Gromoll splitting theorem

Corollary 5: If $Ric \geq 0$, $(\partial_t - \Delta)u = 0$, $u \geq 0$, the

$$\Rightarrow \boxed{(\partial_t - \Delta)|\nabla \ln u|^2 \leq \langle \nabla \log u, \nabla|\nabla \log u|^2\rangle} \rightsquigarrow$$ Li-Yau Harnack inequality